

Implementing the HJM model by Monte Carlo Simulation

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Bob Flagg

Email: bob@calcworks.net

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Abstract

We discuss an implementation of the Heath-Jarrow-Morton model for pricing zero-coupon bonds and interest rate options, including caps and floors, by Monte Carlo simulation. Principal component analysis is used to estimate volatilities of the model from historical time series data for forward rates. Models with as many as 10 factors are supported.

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Chapter 1

Modeling Forward Rates

The Heath-Jarrow-Morton (HJM) model evolves the whole yield curve forward in time. This can be done in several different ways; for example, by modeling the zero-coupon bond yield curve or by modeling the forward rate curve. The HJM model represents the yield curve in terms of forward rates because the no-arbitrage condition has a particularly simple form in this approach.

1.1 Forward Rates

Fix a time horizon T^* (say, 50 years) and assume all bonds considered mature at or before T^* . Let $Z(t, T)$ denote the price at time t of a zero-coupon bond maturing at time T and having face value 1. Recall that the *forward rate* $f(t, T)$ represents the instantaneous continuously compounded rate contracted at time t for riskless borrowing or lending at time $T \geq t$.

To relate $Z(t, T)$ and $f(t, T)$, let δ be a small positive number and consider a portfolio set up at time t with a short position of size 1 in T -maturity zero coupon bonds, which generates income $Z(t, T)$, and a long position of size $\frac{Z(t, T)}{Z(t, T+\delta)}$ in $(T + \delta)$ -maturity zero coupon bonds, which costs $Z(t, T)$. The net cost of setting up this portfolio at time t is zero. At time T , holding this portfolio requires that we pay 1 to cover the short position in T -maturity zero coupon bonds. At time $T + \delta$, we receive $\frac{Z(t, T)}{Z(t, T+\delta)}$ from the long position in the $(T + \delta)$ -maturity zero coupon bonds. That is, we have invested 1 at time T and received $\frac{Z(t, T)}{Z(t, T+\delta)}$ at time $T + \delta$. The yield of this investment is

$$\frac{1}{\delta} \log \frac{Z(t, T)}{Z(t, T + \delta)} = -\frac{\log Z(t, T + \delta) - \log Z(t, T)}{\delta}. \quad (1.1)$$

This is the continuously compounded rate of interest that, applied to 1 at time T will return $\frac{Z(t, T)}{Z(t, T+\delta)}$ at time $T + \delta$.

Based on the above observations, it is natural to define the *forward rate at time*

t for investing at time T as

$$f(t, T) = -\lim_{\delta \downarrow 0} \frac{\log Z(t, T + \delta) - \log Z(t, T)}{\delta} = -\frac{\partial}{\partial T} \log Z(t, T). \quad (1.2)$$

This is the limit of the yield in (1.1) as $\delta \downarrow 0$ and can thus be regarded as the *instantaneous interest rate* at time T that can be locked in at the earlier time t .

If we know $f(t, T)$ for all values $0 \leq t \leq T \leq T^*$, we can recover $Z(t, T)$ by the formula

$$Z(t, T) = e^{-\int_t^T f(t, v) dv}, \quad 0 \leq t \leq T \leq T^*. \quad (1.3)$$

Finally, since the instantaneous rate we can lock in at time t for investing at time t is the short rate, $r(t)$, we have

$$r(t) = f(t, t). \quad (1.4)$$

1.2 Dynamics of Forward Rates and Bond Prices

In the HJM model, the evolution of the forward rate curve is modeled by a stochastic differential equation of the form

$$df(t, T) = \mu(t, T)dt + \sigma(t, T)dW(t), \quad 0 \leq t \leq T, \quad (1.5)$$

where $df(t, T)$ denotes the differential with respect to t with T held constant. In general, W is a standard d -dimensional Brownian motion (d is the number of *factors*) but to simplify the presentation we initially take $d = 1$.

From (1.5), we can work out the dynamics of the bond prices given by (1.3). Using the Leibniz integral rule, we get

$$d\left(-\int_t^T f(t, v)dv\right) = f(t, t)dt - \int_t^T df(t, v)dv.$$

This together with (1.4) gives

$$d\left(-\int_t^T f(t, v)dv\right) = r(t)dt - \int_t^T [\mu(t, v)dt + \sigma(t, v)dW(t)]dv.$$

Let

$$\mu^*(t, T) = \int_t^T \mu(t, v)dv \quad (1.6)$$

and

$$\sigma^*(t, T) = \int_t^T \sigma(t, v)dv \quad (1.7)$$

Then interchanging the order of integration in the expression above yields

$$d\left(-\int_t^T f(t, v)dv\right) = r(t)dt - \mu^*(t, T)dt - \sigma^*(t, T)dW(t). \quad (1.8)$$

A simple application of Itô's formula then gives us

$$dZ(t, T) = Z(t, T) \left[r(t) - \mu^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2 \right] dt - \sigma^*(t, T) Z(t, T) dW(t). \quad (1.9)$$

1.3 The No-Arbitrage Condition

The model we built in the last section has a zero-coupon bond with maturity T for every T with $0 \leq T \leq T^*$. To be sure there is no chance of arbitrage when trading in these bonds, we need to find a measure $\tilde{\mathbf{P}}$ under which discounted zero-coupon bond prices are martingales. Let the discount factors be denoted by $D(t)$ so that

$$D(t) = e^{-\int_0^t r(u) du}.$$

Since $dD(t) = -r(t)D(t)dt$,

$$\begin{aligned} d(D(t)Z(t, T)) &= -r(t)D(t)Z(t, T)dt + D(t)dZ(t, T) \\ &= D(t)Z(t, T) \left[\left(-\mu^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2 \right) dt - \sigma^*(t, T)dW(t) \right] \end{aligned} \quad (1.10)$$

If we can write the term in square brackets as

$$-\sigma^*(t, T)[\Theta(t)dt + dW(t)],$$

then we can use Girsanov's Theorem to change to a measure $\tilde{\mathbf{P}}$ under which

$$\tilde{W}(t) = \int_0^t \Theta(u)du + W(t) \quad (1.11)$$

is a standard Brownian motion. We can then rewrite (1.10) as

$$d(D(t)Z(t, T)) = -D(t)Z(t, T)\sigma^*(t, T)d\tilde{W}(t), \quad (1.12)$$

from which it would follow that $\{D(t)Z(t, T)\}_t$ is a martingale under $\tilde{\mathbf{P}}$. To achieve this goal, we must solve the equation

$$\begin{aligned} \left(-\mu^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2 \right) dt - \sigma^*(t, T)dW(t) \\ = -\sigma^*(t, T)[\Theta(t)dt + dW(t)] \end{aligned}$$

for $\Theta(t)$. Equivalently, we must find a process $\Theta(t)$ satisfying

$$-\mu^*(t, T) + \frac{1}{2}(\sigma^*(t, T))^2 = -\sigma^*(t, T)\Theta(t) \quad (1.13)$$

for all $0 \leq t \leq T \leq T^*$. Note that there are infinitely many equations, one for each $T \leq T^*$; that is, there is one *market price of risk equation* for each maturity. However, there is only one *market price of risk processes* $\Theta(t)$. In the general case, there are d *market price of risk processes* $\Theta_1(t), \dots, \Theta_d(t)$, one for each source of uncertainty.

From (1.6) and (1.7), it follows that

$$\frac{\partial}{\partial T}\mu^*(t, T) = \mu(t, T), \quad \frac{\partial}{\partial T}\sigma^*(t, T) = \sigma(t, T).$$

Differentiating (1.13) with respect T , we get

$$-\mu(t, T) + \sigma^*(t, T)\sigma(t, T) = -\sigma(t, T)\Theta(t)$$

or

$$\mu(t, T) = \sigma(t, T)[\sigma^*(t, T) + \Theta(t)]. \quad (1.14)$$

As long as $\sigma(t, T)$ is nonzero, we can solve (1.14):

$$\Theta(t) = \frac{\mu(t, T)}{\sigma(t, T)} - \sigma^*(t, T), \quad 0 \leq t \leq T. \quad (1.15)$$

1.4 HJM Under the Risk-Free Measure

If we assume the model (1.5) satisfies the no-arbitrage condition (1.14), then it can be rewritten as

$$\begin{aligned} df(t, T) &= \mu(t, T)dt + \sigma(t, T)dW(t) \\ &= \sigma(t, T)\sigma^*(t, T)dt + \sigma(t, T)[\Theta(t)dt + dW(t)] \\ &= \sigma(t, T)\sigma^*(t, T)dt + \sigma(t, T)d\tilde{W}(t). \end{aligned}$$

In other words, to assure that there is no arbitrage, we need the drift of the forward rate under the risk-free measure to be $\sigma(t, T)\sigma^*(t, T)$. In this case we can also rewrite equation (1.10) as

$$\begin{aligned} d(D(t)Z(t, T)) &= \sigma^*(t, T)D(t)Z(t, T)[\Theta(t)dt + dW(t)] \\ &= -\sigma^*(t, T)D(t)Z(t, T)d\tilde{W}(t) \end{aligned}$$

and so the dynamics of a zero-coupon bond becomes

$$\begin{aligned} dZ(t, T) &= d\left(\frac{1}{D(t)}D(t)Z(t, T)\right) \\ &= \frac{r(t)}{D(t)}D(t)Z(t, T)dt - \sigma^*(t, T)\frac{1}{D(t)}D(t)Z(t, T)d\tilde{W}(t) \\ &= r(t)Z(t, T)dt - \sigma^*(t, T)Z(t, T)d\tilde{W}(t). \end{aligned}$$

The derivation of these results for the general d -dimensional case is a simple generalization of what we have done above so we will state them without proofs. The SDE for the d -factor model becomes

$$df(t, T) = \left(\sum_{i=1}^d \sigma_i(t, T) \sigma_i^*(t, T) \right) dt + \sum_{i=1}^d \sigma_i(t, T) d\tilde{W}_i(t) \quad (1.16)$$

and the dynamics of a zero-coupon bond becomes

$$dZ(t, T) = r(t)Z(t, T)dt - Z(t, T) \sum_{i=1}^d \sigma_i^*(t, T) d\tilde{W}_i(t). \quad (1.17)$$

Chapter 2

Implementing the Model

Our implementation of the HJM model uses Monte Carlo to simulate (1.16). However, except for very simple cases of σ , exact simulation of (1.16) is not feasible and one must resort to discrete approximation. We will follow Glasserman ([1], Section 3.6.2) quite closely.

2.1 The Discrete Drift

Discrete approximation of forward rates will require discretization of both arguments of $f(t, T)$. To simplify notation we will fix a time grid $0 = t_0 < t_1 < \dots < t_M$ for the first argument and then for each fixed time t_i we use the subgrid $t_i < t_{i+1} < \dots < t_M$ for discretizing the second argument.

We use $\hat{f}(t_i, t_j)$ to denote the discretized forward rate for maturity t_j as of time t_i , $j \geq i$ and $\hat{Z}(t_i, t_j)$ to denote the corresponding discretized zero-coupon bond price, so that

$$\hat{Z}(t_i, t_j) = \exp\left(-\sum_{l=i}^{j-1} \hat{f}(t_i, t_l)(t_{l+1} - t_l)\right) \quad (2.1)$$

To minimize discretization error, we would like the initial values of the discretized zero-coupon bonds $\hat{Z}(0, t_j)$ to match the exact values $Z(0, t_j)$ for all maturities t_j on the grid. From (1.3) and (2.1) it follows that this holds if

$$\sum_{l=i}^{j-1} \hat{f}(0, t_l)(t_{l+1} - t_l) = \int_0^{t_j} f(0, v) dv;$$

that is, if

$$\hat{f}(0, t_l) = \frac{1}{t_{l+1} - t_l} \int_{t_l}^{t_{l+1}} f(0, v) dv, \quad (2.2)$$

for all $l = 0, 1, \dots, M-1$. So we should initialize $\hat{f}(0, t_l)$ to the average level of the forward curve $f(0, T)$ on the interval $[t_l, t_{l+1}]$.

Once the initial forward rate curve has been specified, a simulation of a single-factor model evolves according to

$$\begin{aligned} \hat{f}(t_i, t_j) &= \hat{f}(t_{i-1}, t_j) + \\ &\hat{\mu}(t_{i-1}, t_j)[t_i - t_{i-1}] + \hat{\sigma}(t_{i-1}, t_j)\sqrt{t_i - t_{i-1}}Z_i, \quad j = i, \dots, M, \end{aligned} \quad (2.3)$$

where Z_1, \dots, Z_M are independent standard normal random variables and $\hat{\mu}$ and $\hat{\sigma}$ are the discrete counterparts of the continuous-time coefficients in (1.5).

We will specify the discrete volatility $\hat{\sigma}$ by calibrating from market data in the next section. To choose the discrete drift $\hat{\mu}$ to approximate the no-arbitrage condition

$$\mu(t, T) = \sigma(t, T) \int_t^T \sigma(t, v) dv, \quad (2.4)$$

we exploit the martingale property for discounted bond prices. This property requires that

$$Z(t, T)e^{-\int_0^t f(s, s) ds}$$

be a martingale in t for each T . The corresponding discretized requirement is that

$$\hat{Z}(t_i, t_j)e^{-\sum_{k=0}^{i-1} \hat{f}(t_k, t_k)[t_{k+1} - t_k]}$$

be a martingale in i for each j . This leads (see Glasserman [1] Section 3.6.2) to the discrete version of the HJM no arbitrage condition:

$$\begin{aligned} \hat{\mu}(t_{i-1}, t_j)[t_{j+1} - t_j] &= \\ \frac{1}{2} \left(\sum_{l=i}^j \hat{\sigma}(t_{i-1}, t_l)[t_{l+1} - t_l] \right)^2 &- \frac{1}{2} \left(\sum_{l=i}^{j-1} \hat{\sigma}(t_{i-1}, t_l)[t_{l+1} - t_l] \right)^2. \end{aligned}$$

The d -factor version of this condition is

$$\begin{aligned} \hat{\mu}(t_{i-1}, t_j)[t_{j+1} - t_j] &= \\ \frac{1}{2} \sum_{k=1}^d \left[\left(\sum_{l=i}^j \hat{\sigma}_k(t_{i-1}, t_l)[t_{l+1} - t_l] \right)^2 \right. &- \left. \left(\sum_{l=i}^{j-1} \hat{\sigma}_k(t_{i-1}, t_l)[t_{l+1} - t_l] \right)^2 \right]. \end{aligned} \quad (2.5)$$

2.2 Calibrating the HJM Model

Because the diffusion process is the same in the dynamics of forward rates under the real world measure as it is in the dynamics under the risk-free measure, we can use historical data to estimate the diffusion, $\sigma(t, T)$, in the HJM model (1.16). We outline informally how to do this calibration of the HJM model following the presentation of Shreve [2] Section 10.3.6.

Assume that $\sigma(t, T)$ is of the form

$$\sigma(t, T) = \tilde{\sigma}(T - t) \quad (2.6)$$

for some deterministic function $\tilde{\sigma}(\tau)$. We will choose $\tilde{\sigma}(\tau)$ to match historical data. The forward rate evolves according to the SDE

$$df(t, T) = \mu(t, T)dt + \tilde{\sigma}(T - t)dW(t), \quad 0 \leq t \leq T.$$

Suppose we have observed this forward rate at times $t_1 < t_2 < \dots < t_J < 0$ in the past and the forward rate we observed at those times was for the relative maturities $\tau_1 < \tau_2 < \dots < \tau_K$; that is, we have observed $f(t_j, t_j + \tau_k)$ for $j = 1, \dots, J$ and $k = 1, \dots, K$. Suppose further that for some small positive δ we have also observed $f(t_j + \delta, t_j + \tau_k)$, where δ is sufficiently small that $t_j + \delta < t_{j+1}$ for $j = 1, \dots, J - 1$ and $t_J + \delta < 0$. According to our model,

$$f(t_j + \delta, t_j + \tau_k) - f(t_j, t_j + \tau_k) \approx \delta\mu(t_j, t_j + \tau_k) + \tilde{\sigma}(\tau_k)(W(t_j + \delta) - W(t_j)).$$

Let

$$D_{j,k} = \frac{f(t_j + \delta, t_j + \tau_k) - f(t_j, t_j + \tau_k)}{\sqrt{\delta}} \quad (2.7)$$

and note that

$$D_{j,k} \approx \sqrt{\delta}\mu(t_j, t_j + \tau_k) + \tilde{\sigma}(\tau_k)\frac{W(t_j + \delta) - W(t_j)}{\sqrt{\delta}}.$$

Since the first term in this equation contains $\sqrt{\delta}$, it is small relative to the second term. So, for

$$X_j = \frac{W(t_j + \delta) - W(t_j)}{\sqrt{\delta}}, \quad j = 1, \dots, J, \quad (2.8)$$

which is a standard normal random variable. we have

$$D_{j,k} \approx \tilde{\sigma}(\tau_k)X_j. \quad (2.9)$$

But the X_1, \dots, X_J are not only standard normal random variables, they are also independent. This means we can regard $D_{1,k}, \dots, D_{J,k}$ as independent observations taken at times t_1, \dots, t_J on forward rates, all with the same relative maturity τ_k . The empirical covariance is

$$C_{k_1, k_2} = \frac{1}{J} \sum_{j=1}^J D_{j, k_1} D_{j, k_2}.$$

The theoretical covariance, computed from the right-hand side of (2.9), is

$$\mathbf{E}[\tilde{\sigma}(\tau_{k_1})\tilde{\sigma}(\tau_{k_2})X_j^2] = \tilde{\sigma}(\tau_{k_1})\tilde{\sigma}(\tau_{k_2}).$$

We would like to find $\tilde{\sigma}(\tau_1), \dots, \tilde{\sigma}(\tau_K)$ so that

$$C_{k_1, k_2} = \tilde{\sigma}(\tau_{k_1})\tilde{\sigma}(\tau_{k_2}), \quad k_1, k_2 = 1, 2, \dots, K. \quad (2.10)$$

But this is (because of symmetry) $\frac{1}{2}K(K+1)$ equations and only K unknowns.

To determine a best choice of $\tilde{\sigma}(\tau_1), \dots, \tilde{\sigma}(\tau_K)$, we use principal component analysis. Let

$$D = \begin{bmatrix} D_{11} & D_{12} & \dots & D_{1K} \\ D_{21} & D_{22} & \dots & D_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ D_{J1} & D_{J2} & \dots & D_{JK} \end{bmatrix}$$

Then the empirical covariance matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1K} \\ C_{21} & C_{22} & \dots & C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ C_{K1} & C_{K2} & \dots & C_{KK} \end{bmatrix} = \frac{1}{J} D^T D$$

is symmetric and positive semidefinite and so has a principal component decomposition

$$C = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \dots + \lambda_K e_K e_K^T,$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K \geq 0$ are the eigenvalues of C and the column vectors e_1, e_2, \dots, e_K are the orthonormal eigenvectors.

As noted above, we would like to find $\tilde{\sigma}(\tau_1), \dots, \tilde{\sigma}(\tau_K)$ so that

$$C = \begin{bmatrix} \tilde{\sigma}(\tau_1) \\ \tilde{\sigma}(\tau_2) \\ \vdots \\ \tilde{\sigma}(\tau_K) \end{bmatrix} \begin{bmatrix} \tilde{\sigma}(\tau_1) & \tilde{\sigma}(\tau_2) & \dots & \tilde{\sigma}(\tau_K) \end{bmatrix}$$

but this may not be possible to do exactly. The best approximation is

$$\begin{bmatrix} \tilde{\sigma}(\tau_1) \\ \tilde{\sigma}(\tau_2) \\ \vdots \\ \tilde{\sigma}(\tau_K) \end{bmatrix} = \sqrt{\lambda_1} e_1.$$

If there are d Brownian motions driving the SDE modeling forward rates, then each component has its own $\tilde{\sigma}$ vector and the best approximation for them is given by

$$\sqrt{\lambda_1} e_1, \dots, \sqrt{\lambda_d} e_d.$$

2.3 The Pricing Algorithm

Once the volatility $\hat{\sigma}$ has been calibrated from market data and the discrete form of the drift has been identified, specifying the details of the HJM pricing algorithm is mostly about keeping track of indices. The description below of the pricing algorithm follows Glasserman [1], Section 3.6.3, very closely.

In implementing the simulation, we are not interested in keeping track of an $M \times M$ matrix of rates as the notation $\hat{f}(t_i, t_j)$ might suggest. At each step, we need only the vector of current rates. Recall that our time and maturity grid consists of the set of dates $0 = t_0 < t_1 < \dots < t_M$, where $t_M = T^*$ is our time horizon, so t_M is the maturity of the longest-maturity bond represented in the model. This means that the last forward rate relevant to the model applies to the interval $[t_{M-1}, t_M]$, the forward rate with maturity argument t_{M-1} . Thus, our initial vector of forward rates consists of the M components $\hat{f}(0, 0), \hat{f}(0, t_1), \dots, \hat{f}(0, t_{M-1})$. These values will be set by (2.2):

$$\hat{f}(0, t_l) = \frac{1}{t_{l+1} - t_l} \int_{t_l}^{t_{l+1}} f(0, v) dv.$$

As the simulation evolves, the number of relevant rates decreases: at time t_i we are only interested in the rates $\hat{f}(t_i, t_i), \hat{f}(t_i, t_{i+1}), \dots, \hat{f}(t_i, t_{M-1})$. We represent these $M - i$ rates remaining at time t_i as the vector (f_1, \dots, f_{M-i}) . Thus we are indexing forward rates by *relative* maturity, as done in the Musiela parametrization.

Similar notational conventions will be used for $\hat{\mu}(t_i, t_j)$ and $\hat{\sigma}_k(t_i, t_j)$, $k = 1, \dots, d$: we use m_j for $\hat{\mu}$ and $s_j(k)$ for $\hat{\sigma}_k$. Thus the simulation step from t_{i-1} to t_i becomes

$$f_j \leftarrow f_{j+1} + m_j[t_i - t_{i-1}] + \sum_{k=1}^d s_j(k) \sqrt{t_i - t_{i-1}} Z_{ik}, \quad j = 1, \dots, M - i,$$

where

$$m_j = \hat{\mu}(t_{i-1}, t_{i+j-1}), \quad s_j(k) = \hat{\sigma}_k(t_{i-1}, t_{i+j-1}). \quad (2.11)$$

Our implementation of the HJM simulation is broken into two main parts: one calculating the discrete drift parameter at a fixed time step and the other looping over time steps and updating the forward curve at each step.

Calculating the discrete drift parameter amounts to evaluating

$$\hat{\mu}(t_{i-1}, t_{i+j-1}) = \frac{1}{2h_j} \left[\sum_{k=1}^d \left(\sum_{l=i}^j \hat{\sigma}_k(t_{i-1}, t_l) h_{l+1} \right)^2 - \sum_{k=1}^d \left(\sum_{l=i}^{j-1} \hat{\sigma}_k(t_{i-1}, t_l) h_{l+1} \right)^2 \right].$$

Algorithm 2.3.1 achieves this in a way that avoids duplicate computation. In the notation of the algorithm, the drift parameter is evaluated as

$$\frac{1}{2(t_{j+1} - t_j)} [B_{\text{next}} - B_{\text{prev}}],$$

and each $A_{\text{next}}(k)$ records the quantity

$$\sum_{l=i}^j \hat{\sigma}_k(t_{i-1}, t_l) h_{l+1}.$$

The inputs are the volatilities $\mathbf{s} = \{s_j(k) : j = 1, \dots, M - i, k = 1, \dots, d\}$, the intervals $\mathbf{h} = \{h_l : l = 1, \dots, M\}$ (where $h_l = t_l - t_{l-1}$), and the step index i .

Algorithm 2.3.1: UPDATEDRIFT($\mathbf{s}, \mathbf{h}, i$)

```

 $A_{\text{prev}}(k) \leftarrow 0, k = 1, \dots, d;$ 
for  $j \leftarrow 1$  to  $M - i$ 
   $B_{\text{next}} \leftarrow 0;$ 
  for  $k \leftarrow 1$  to  $d$ 
    do  $\begin{cases} A_{\text{next}}(k) \leftarrow A_{\text{prev}}(k) + s_j(k) \times h_{i+j}; \\ B_{\text{next}} \leftarrow B_{\text{next}} + A_{\text{next}}(k) \times A_{\text{next}}(k); \\ A_{\text{prev}}(k) \leftarrow A_{\text{next}}(k); \end{cases}$ 
   $m_j \leftarrow (B_{\text{next}} - B_{\text{prev}})/(2h_{i+j});$ 
   $B_{\text{prev}} \leftarrow B_{\text{next}};$ 
return  $(m_1, \dots, m_{M-i})$ 

```

Algorithm 2.3.2 implements a single replication of the pricing procedure in an HJM simulation, which is repeated many times to estimate a product price. This algorithm calls 2.3.1 to calculate the discrete drift for all remaining maturities at each time step. The key step in the algorithm used to value interest rate derivatives is

$$P \leftarrow \text{cashflow at } t_i,$$

where, of course, the particular instrument determining the cashflow at t_i must be provided as input. The remaining inputs are the initial curve $\mathbf{f} = \{f_l : l =$

$1, \dots, M\}$ and the intervals $\mathbf{h} = \{h_l : l = 1, \dots, M\}$.

Algorithm 2.3.2: PRESENTVALUE($\mathbf{f}, \mathbf{h}, \text{product}$)

```

 $D \leftarrow 1; P \leftarrow 0; C \leftarrow 0;$ 
for  $i \leftarrow 1$  to  $M - 1$ 
     $D \leftarrow D \times e^{-f_1 \times h_i};$ 
    for  $j \leftarrow 1$  to  $M - i$ 
        do  $\left\{ \begin{array}{l} \textbf{for } k \leftarrow 1 \textbf{ to } d : s_j(k) \leftarrow \hat{\sigma}_k(t_{i-1}, t_{i+j-1}); \\ (m_1, \dots, m_{M-i}) \leftarrow \text{driftParameters}(\mathbf{s}, \mathbf{h}, i); \\ \text{generate } Z_1, \dots, Z_d \sim \mathcal{N}(0, 1); \end{array} \right.$ 
    do  $\left\{ \begin{array}{l} \textbf{for } j \leftarrow 1 \textbf{ to } M - i \\ \quad \left\{ \begin{array}{l} S \leftarrow 0; \\ \textbf{do } \left\{ \begin{array}{l} \textbf{for } k \leftarrow 1 \textbf{ to } d : S \leftarrow S + s_j(k) \times Z_k; \\ f_j \leftarrow f_{j+1} + m_j * h_i + S \times \sqrt{h_i}; \end{array} \right. \\ P \leftarrow \text{cashflow at } t_i(\text{depending on the instrument}); \\ C \leftarrow C + D \times P; \end{array} \right.$ 
return ( $C$ )

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Chapter 3

Numerical Results

In this chapter we describe the results of applying our implementation of the HJM model to pricing some simple fixed-income products, including zero-coupon bonds, caps and floors.

3.1 Zero-Coupon Bonds

There is no reason to use HJM simulation to price bonds: the simulation will simply return prices that could have been computed from the initial forward curve, using the formula:

$$Z(0, t_j) \approx \hat{Z}(0, t_j) = e^{-\sum_{l=0}^{j-1} \hat{f}(0, t_l)(t_{l+1}-t_l)}.$$

However, this provides one way to test the implementation. The connection between the simulated approximation to zero-coupon prices and true prices can be made precise as follows. Consider a simulating path \hat{f} . From this we can extract discretized short rates:

$$\hat{r}(t_0) = \hat{f}(t_0, t_0), \hat{r}(t_1) = \hat{f}(t_1, t_1), \dots, \hat{r}(t_M) = \hat{f}(t_M, t_M).$$

From these we can calculate a discount factor

$$\hat{D}(t_j) = e^{-\sum_{l=0}^{j-1} \hat{r}(t_l)(t_{l+1}-t_l)} \quad (3.1)$$

for each maturity t_j . Imagine repeating this over n independent paths and let $\hat{D}^{(1)}(t_j), \dots, \hat{D}^{(n)}(t_j)$ denote discount factors calculated over the n paths. It follows from the strong law of large numbers, the martingale property and (2.2), that, almost surely,

$$\frac{1}{n} \sum_{i=1}^n \hat{D}^{(i)}(t_j) \rightarrow \mathbf{E}(\hat{D}(t_j)) = \hat{Z}(0, t_j).$$

This means that if we use our simulation to price a bond, then the results should converge to the value to which the model was initially calibrated. The table below compares simulated and “analytic” prices for several zero-coupon bonds with face value of \$1.00. All results agree to within about 1 cent.

Maturity	$Z(0, T)$ Analytic	$Z(0, T)$ Simulated
1 Month	0.9996	0.9996
1 Quarter	0.9987	0.9987
1 Year	0.9941	0.9941
2 Year	0.9832	0.9831
5 Year	0.9034	0.9022
10 Year	0.7077	0.6968

3.2 Caps

A *caplet* is an interest rate derivative providing protection against an increase in an interest rate for a single period. A *cap* is a portfolio of caplets covering multiple periods. The caplet is like a call option on the short rate, having a payoff of the form $(r(T) - R)^+$ for maturity T and fixed rate R . The underlying rate in a caplet applies over an interval and is based on discrete compounding. To simplify the implementation, we will assume the interval has the form $[t_i, t_{i+1}]$. At t_i , the continuously compounded rate for this interval is $\hat{f}(t_i, t_i)$; the corresponding discretely compounded rate \hat{F} satisfies

$$\frac{1}{1 + \hat{F}(t_i)[t_{i+1} - t_i]} = e^{-\hat{f}(t_i, t_i)[t_{i+1} - t_i]},$$

so that,

$$\hat{F}(t_i) = \frac{1}{t_{i+1} - t_i} \left(e^{-\hat{f}(t_i, t_i)[t_{i+1} - t_i]} - 1 \right).$$

The payoff of the caplet would then be $(\hat{F}(t_i) - R)^+$ (times the principal). The payment is made at the end of the interval, t_{i+1} , and so must be discounted back to t_i , which (using the notation of Algorithm 2.3.2) leads to

$$P \leftarrow e^{-f_1 h_{i+1}} \left(\frac{1}{h_{i+1}} (e^{f_1 h_{i+1}} - 1) - R \right)^+.$$

Some cap prices computed using our implementation are given below. These all have tenor one quarter and interest rate 0.4%. We use such a small fixed rate because the data we use to calibrate the model has very low forward rates early on in the most recent period.

Maturity	Simulated Cap Price
1 Quarter	0.0012
6 Month	0.0029
1 Year	0.0085
5 Year	0.3438
10 Year	1.1687

Evaluating floors in the HJM model is a trivial modification of the methods used for caps so we omit any discussion of them.

Appendix A

Using the Software

Our implementation of the HJM model is done in Java 6 and distributed as a jar archive named **CQF-HJM.jar**. The jar file is available in the directory **HJM** on the included CD. That directory contains a subdirectory **lib** which contains Java libraries required by the application. The main source code is also included on the CD in a subdirectory of **HJM** called **src**. Additional source code used by the HJM application but not specific to that project is included on the CD in a separate folder call **CQF**.

A.1 Using the Java Application

To start the application, right click on the archive **CQF-HJM.jar** and choose *Open with Java 6 Runtime*.¹

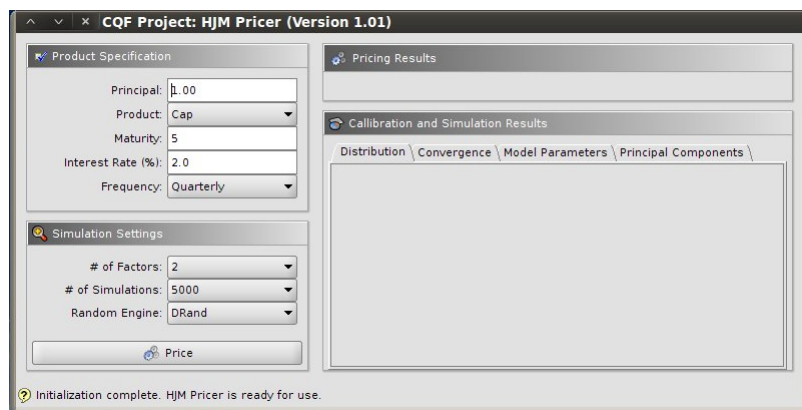


Figure A.1: HJM Main Application Frame

¹This assumes you have Java 6 installed on your machine. If this is not the case, it can be obtained from Oracle at <http://www.java.com/en/download/index.jsp>.

When the application is launched, a main panel is displayed as shown in Figure A.1. The two panels on the left side of the main frame allow you to specify the properties of the fixed-income product you would like to price, the number of factors you want in the HJM model, the number of steps to run in the Monte Carlo simulation and which random engine you would like to use in the simulation. After setting the specifications, click the **Price** button at the bottom left of the main frame. When the calibration and simulation are complete, the price will be shown in the top right subpanel and the tabs of the panel on the bottom right show visualizations of the distribution of approximations to the price during the simulation, the convergence of these approximations and the principal components of the model. The model parameters are also shown in one of the tabs. Examples of these visualizations are shown below. To quit the application, close the window.

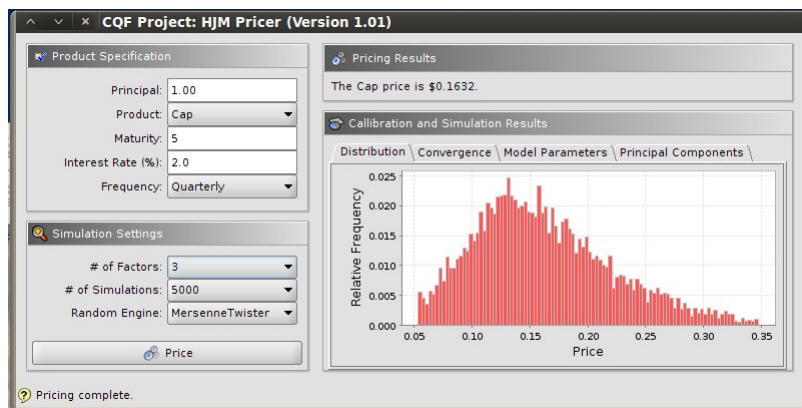


Figure A.2: HJM Distribution Visualization

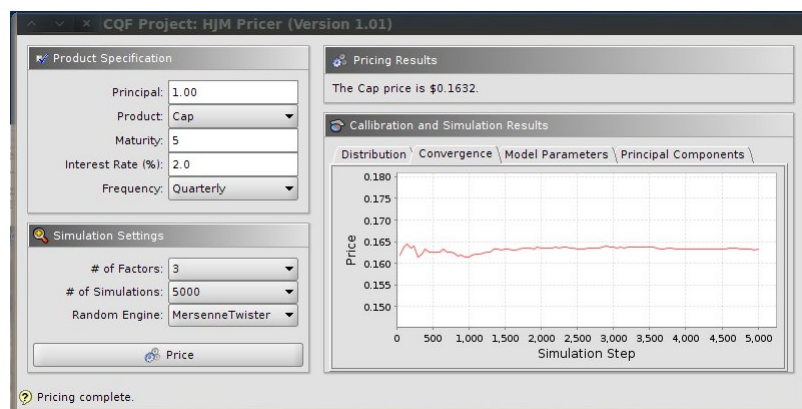


Figure A.3: HJM Convergence Visualization

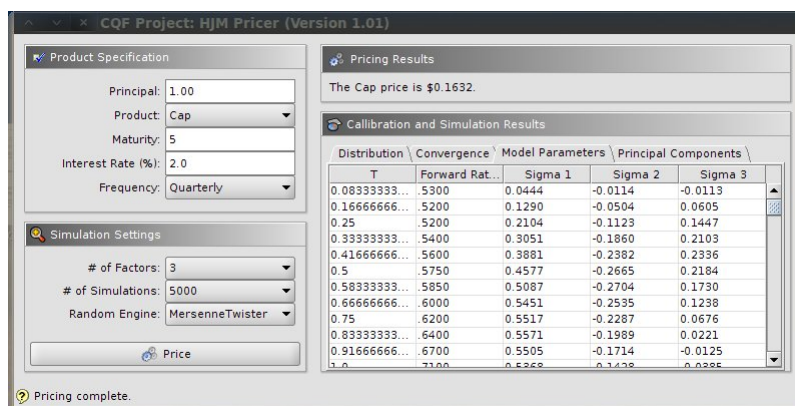


Figure A.4: HJM Parameters Table

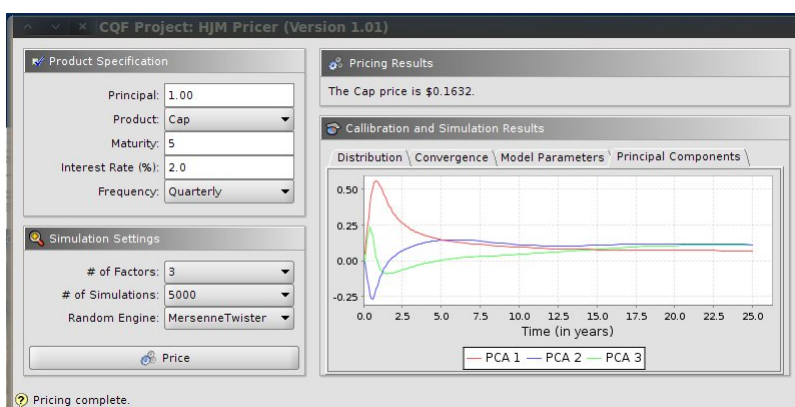


Figure A.5: HJM PCA Visualization

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